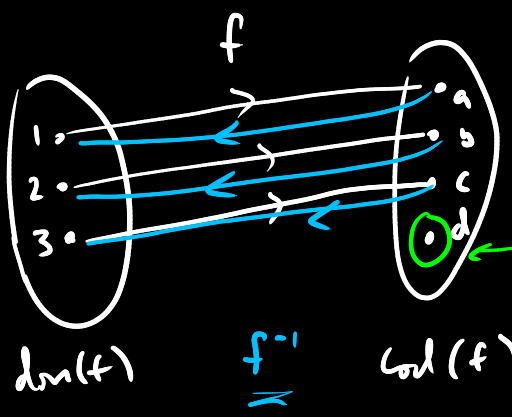
Last Time: More on Functions

Wrapping Up Functions:

① $f^{-1}(S) = \{x \in dom(f) : f(x) \in S\}$.

NB: Sometimes $f^{-1}\{x\}$ is single valued...
If all $f^{-1}\{x\}$ are single-valued, then
f^{-1} can be thought of as a function...



$f^{-1}\{d\} = \emptyset$,
but we could define
$g(d) = 1$ or $g(d) = 2$
or $g(d) = 3$

we would define

$$g(x) = \begin{cases} y & \text{if } f^{-1}\{x\} = \{y\} \\ 1 & \text{o/w.} \end{cases}$$

with this definition of g,

$$(g \circ f)(x) = g(f(x)) = x \quad \text{for all } x \in dom(f).$$
(f is injective)

Point: $g \circ f = id_{dom(f)}$ for f and g above...

Defn: A function f has...

① left inverse $g : cod(f) \to dom(f)$ when $g \circ f = id_{dom(f)}$.

② right inverse $g : cod(f) \to dom(f)$ when $f \circ g = id_{cod(f)}$.

③ two-sided inverse $g : cod(f) \to dom(f)$ when g is both a left and right inverse.

Prop: Let f be a function.

① f is injective if and only if f has a left inverse.

② f is surjective if and only if f has a right inverse.

③ f is bijective if and only if f has a two-sided inverse.

Exercise: Prove this proposition (assuming $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ are both finite).

Final Comments:

Prop: If f has both a left inverse and a right inverse, then f has a unique two-sided inverse; indeed, the left and right inverses given were equal!

$$\left(g \circ f = \mathrm{id}_{\mathrm{dom}(f)} \text{ and } f \circ h = \mathrm{id}_{\mathrm{cod}(f)} \longrightarrow g = h \right).$$

→ Again, very good exercise!

When f is only injective (resp surjective), f can have many left (resp right) inverses...

Graph Theory

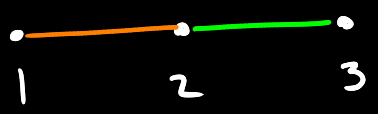
Defn: A (simple) graph is a pair $G = (V, E)$ satisfying V is a set, and $E \subseteq \{e \subseteq V : \#e = 2\}$.

The elements of V are vertices of G, and elements of E are edges of G.

Remark: The singular form "vertices" is "vertex".

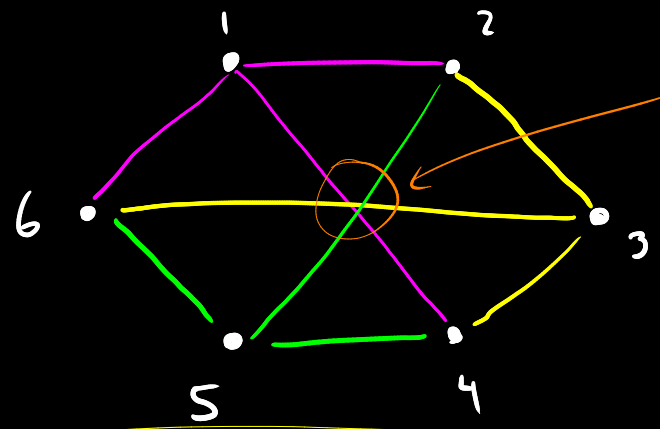
Please don't write ~~"vertice"~~ or ~~"vertexes"~~

Ex: The pair $G = ([3], \{\{1,2\}, \{2,3\}\})$ is a graph.



Ex: The pair $G = \left([6], \left\{ \begin{matrix} \{1,2\}, & \{1,4\}, & \{1,6\}, \\ \{3,2\}, & \{3,4\}, & \{3,6\}, \\ \{5,2\}, & \{5,4\}, & \{5,6\} \end{matrix} \right\}\right)$

is a graph.



those crossings are NOT a vertex...

Fun Fact: This graph will always have a crossing in pictures... try it...

$[1 \quad 2]$ ← "empty graph"

NonEx: The following are NOT Graphs...

$G = ([4], \{1\})$ ← not a pair, just an elt

$H = ([4], \{\{1,2\}, \{1\}\})$ ← not a pair... singleton.

Not a graph → $J = ([4], \{1,2\})$ | Point: $S \neq \{S\}$

↑ pair of vertices, not a set of pairs...

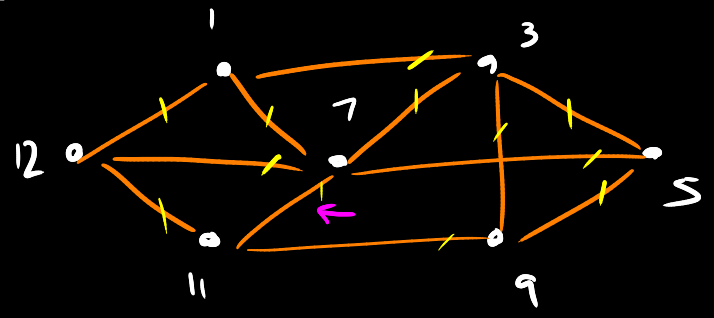
$K = ([4], \{\{1,2\}, \{1,3\}, \{4,5\}, \{3,4\}\})$.

↑ 5 is not a vertex...

Ex: $G = ([4], \emptyset)$ is a graph...

← picture of G

Ex: The graph depicted by



G

Vertices of G ↓

has $V(G) = \{1, 3, 5, 7, 9, 11, 12\}$

and $E(G) = \{\{1,3\}, \{1,7\}, \{1,12\}, \{3,5\}, \{3,7\}, \{3,9\}, \{5,7\}, \{5,9\}, \{7,11\}, \{7,12\}, \{9,11\}, \{11,12\}\}$

So $G = (V(G), E(G))$.

Defn: Let G be a graph. Vertices $u, v \in V(G)$ are <u>adjacent</u> when there is an edge connecting u to v.

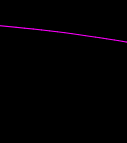
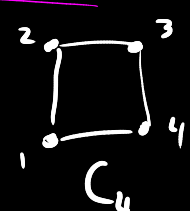
A vertex v is <u>incident</u> to edge e when v is an <u>end</u> of e.

u is adjacent to v

u, v are incident to e

u, v are ends of e.

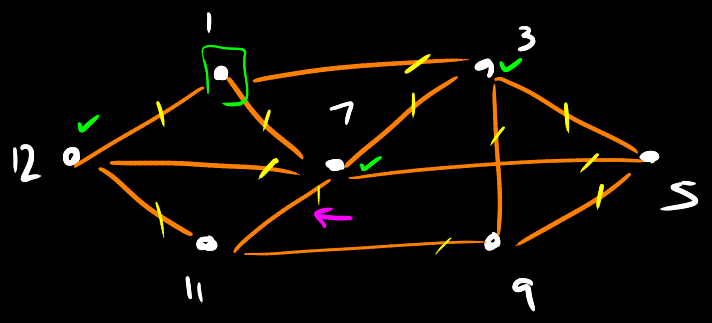
Special Classes of Graphs

Name	Notation	Definition	
Complete Graph	K_n	$V(K_n) = [n]$ $E(K_n) = \{\{i,j\} : 1 \le i < j \le n\}$	
Empty Graph	$\overline{K}_n$	$V(\overline{K}_n) = [n]$ $E(\overline{K}_n) = \emptyset$	
Cycle $(n \ge 3)$	C_n	$V(C_n) = [n]$ $E(C_n) = \{\{i, i+1\} : i \in [n-1]\} \cup \{\{1, n\}\}$	
Path Graph	P_n	$V(P_n) = [n+1]$ $E(P_n) = \{\{i, i+1\} : i \in [n]\}$	

Defn: Let G be a graph and $v \in V(G)$. The <u>neighborhood</u> of v in G is $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v\}$.

The <u>degree</u> of v is $\deg_G(v) = \#N_G(v)$.

Ex: The graph G_1 depicted here has neighborhoods and degrees given below:



G_1

$N_{G_1}(1) = \{3, 7, 12\}$ $\quad \deg_{G_1}(1) = 3$

$N_{G_1}(3) = \{5, 7, 9, 1\}$ $\quad \deg_{G_1}(3) = 4$

Defn: The degree sequence of graph G_1 is the sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where $d_1, d_2, \ldots, d_n$ are the degrees of all vertices of G_1.

Exercise: Write down degree sequence of the preceding example.